PKU-HUST Lectures (2020-2021)

OV

→ 'Modeling Dynamical Systems', for MFE Applications (c.f. after Biophysics)

→ a useful (I hope) methods

mini-course (minimal practical knowledge)

→ Emphasis on { nonlinear dynamics, space-time scales } ⇒ MFE

→ Topics:

i.) Feedback loops, predator-prey (time)

ii.) transition evolution and propagation (space-time) Fisher, FN

iii.) avalanches, traffic flow, bursty flux (Burgers; → kdv/solitons)

iv.) nonlinear diffusion and spreading.

v.) Negative viscosity and Cahn-Hilliard eqn vi.) Model reduction and renormalization – a look at the theory

+

History of Nonlinear Plasma Theory (1 → 2 lectures)

E P ~~S~~ U H

P.D., Frisch, Y. Pomeau

Sources:

- J. D. Murray, "Mathematical Biology"
- R. May, "Stability and Complexity in Model Ecosystems"
- F. Morrison, "The Art of Modelling Dynamic Systems"

+

more coming

- selected papers.

PKU-HUST Lectures I

(i) Feedback Loops and Predator-Prey Systems

Plan:

- OV and Motivation { Drift Wave + Shear Flow
  system

- { Constructing / Solving } the System - { Very Simple Examples

- Lessons from Ecology (c.f. May)
  → General structure; constraint
  → Aside: Bigger Models, Stability + Complexity
  General structure &
  → Kolmogorov Theorem; → Sensible structure
  { Fixed pts. vs. Cycles
  { Feedback
  → Time Delays → Cycles

- Fluctuating Environments and statistical Approaches.
  - → Physics Motivation
  - → Fluctuating Logistics - a simple example.
  - → Applications

O D
↓
Fronts

- Looking Ahead.

I) QV and Basics

→ Fundamental Problem(s)

- Drift waves + zonal shear flow
- L → H transition

- Natural candidate for Predator - Prey Model

2 animal

Hares

- what?:

$$\frac{dH}{dt} = H\,F(H,P)$$

rates

$$\frac{dP}{dt} = P\,G(H,P)$$

$P(t)$, $H(t)$

$H \equiv$ prey

$P \equiv$ predator

(OD)

can extend to 2m x 2m.

Lotka - Volterra

- What does it mean?:

$H \rightarrow \boxed{\mathcal{E}}$ - fluctuation intensity, energy

$P \rightarrow U = V'^2$ - mean square $V_E'^2$

OD ?

$\mathcal{E} \equiv \langle \mathcal{E} \rangle$

$V'^2 = \langle V'^2 \rangle$

} avg. over thin layer [ignores spatial evolution]

- why?

2 populations

$\langle \tilde{U} \rangle$

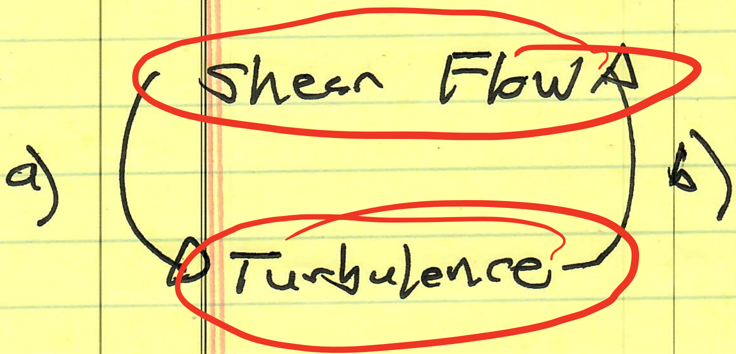
$\mathcal{E}$

→ self-regulation (feedback)

→ symmetry constraints

constructed

1) self-regulation



a) shear stabilizes, controls turbulence
→ everyone knows ....

b) turbulence drives flow, via
Reynolds stress

$$\boxed{\partial_t \langle V_\perp \rangle = -\nabla_\perp \langle \tilde{v}_r \tilde{v}_\theta \rangle}$$

→ $\underline{\nabla \cdot J = 0}$

$\underline{J} = J_{pol} + J_{ps} + J_{||}$

n.b. more precisely:
from vorticity eqn. $(\nabla \cdot J = 0)$

Design $\Rightarrow$

$$\partial_t \left(\boxed{\rho_s^2} \, \partial_r^2 \langle \phi \rangle \right) = -\partial_r \langle \tilde{v}_r \rho_s^2 \nabla_\perp^2 \tilde{\phi} \rangle + \ldots$$

net polarization charge

inhomogeneity

flux of polarization charge

$\langle \tilde{v}_r \tilde{n}_{i_{gc}} \rangle - \langle \tilde{v}_r \tilde{n}_e \rangle$

(ρ_s generalizes – Rosenbluth Hinton 97)

↓

E×B shear flow

NOTE

Reynolds force

$\Downarrow$

1 symm.

but: $\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle = \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle$

HW: show this!

(G.I.) Taylor identity (100+ yrs old)

Upshot:

- Reynolds stress, force $\leftrightarrow$ polarization charge flux

* - Thm: McIntyre + Wood: R. Wood

"PV mixing" [i.e. polarization charge flux] + 1 direction of symmetry

$\Rightarrow$ Z. F. generation.

$\therefore$ How reconcile? $\Rightarrow$ feedback loop

'05
O.G. 2014
2019 Hahm

2) Symmetry* (already encountered)

Why is ZF 'special'? (P.D., I², H '05)

ZF is mode of: ($k_r \neq 0$, $n, m \to 0$)

- minimal inertia - easily excited

$k_\parallel \to 0$ — HW

$(1+k_\perp^2\rho^2) \Rightarrow k_\perp^2\rho^2$ *

$\Downarrow$ H-M $\Downarrow$ 2D fluid

- minimal transport (zero) (none!) *

- $n = 0$, $m = 0$ $\Rightarrow$ $\tilde{V}_r = 0$

can't relax ∇n, ∇T, etc.

- minimal damping *

(no Landau damping - R H, '97)

Symmetry → P-P.

⇒ Symmetry constraints + consideration of self-regulation

⇒ Z. F. can grow / be excited only via 'feeding off' of turbulence, yet shears turbulence, thus regulating it.

⇒ $\boxed{\text{natural problem for predator-prey formulation!}}$

→ What does it look like? $E^{1+\alpha}$
(cf. PD, et. al. '94) Mixing Length process

$$\frac{1}{2}\frac{dE}{dt} = \gamma_0 E - \alpha_1 E^2 - \alpha_2 U E$$

growth turb. — self satn. — coupling key

$$\frac{1}{2}\frac{dU}{dt} = -\mu_1 U + \alpha_2 U E$$

Flow damping

$\langle \tilde{v} \rangle^2$

→ see table in paper for coeffs

Fluctns

8a.

→ N.B. – simplest

→ An infinity of extensions:

most must.

~ $\gamma_0 \rightarrow \bar{\gamma}_0(\nabla p) \rightarrow \bar{\gamma}_0(a/L_p - a/L_{p\,crit})$

coupling to profile evolution

$$\partial_t \langle p \rangle + \partial_r \langle \tilde{v}_r \tilde{p} \rangle = nQ = n(\bar{Q} + \tilde{Q})$$

noise

$$\partial_t \langle p \rangle + \partial_r \chi_0 \mathcal{E} \partial_r \langle p \rangle = n(\bar{Q} + \tilde{Q})$$

0D $\quad \partial_t \langle p \rangle + D_0 \mathcal{E} \langle p \rangle = n(\bar{Q} + \tilde{Q})$

Malkov, P.D. 2009 → $\boxed{\text{Kim, P.D. '03}}$

~ $V_E = \frac{\nabla P}{n} + V_0$ (Carreras, Malkov)

~ 1D, toroidal flow, particles, stoch. field .....

Users guide to Ared - Ares extension

$u = \langle V_\perp \rangle^2$

B.) $\boxed{\text{How is it Constructed ?}}$

Key term: $\boxed{\alpha_2 U E}$ → Reynolds Coupling

Point: Reynolds work transfers energy flow ⟷ fluctuations

How: 2 ways

Now, $\partial_t \langle V_\perp \rangle = - \partial_r \langle \tilde{v}_r \tilde{v}_\perp \rangle$

$$\underbrace{\partial_t \int dr \frac{\langle V_\perp \rangle^2}{2}}_{} = \underbrace{\int \langle \tilde{v}_r \tilde{v}_\perp \rangle \partial_r \langle V_\perp \rangle}_{}$$

but

$$\partial_t \underset{\text{fluctu}}{\mathcal{E}_K} + \partial_t \int dr \frac{\langle V_\perp \rangle^2}{2} = 0$$

layer

so $$\boxed{\partial_t \mathcal{E}_K = - \int dr \, \underline{\langle \tilde{v}_r \tilde{v}_\perp \rangle} \, \underline{\partial_r \langle V_\perp \rangle}}$$

Formally $\partial_t \langle \mathcal{E}_K \rangle = - \langle \tilde{v}_r \tilde{v}_\perp \rangle \langle V_\perp \rangle'$

Layer average

What of stress?:

→ 'Shear — Eddy Tilting Feedback Loop'

$$\langle \tilde{v}_r \tilde{v}_\perp \rangle = -\frac{c^2}{B^2} \sum_n k_x k_y |\phi_n|^2$$ Zip

~ $\langle k_x k_y \rangle$ → eddy orientation!

i.e. $|\phi_n|^2 = f(k_x/\bar{k}_x)\, g(k_y/\bar{k}_y)$

f, g even functions

$\langle k_x k_y \rangle = 0$ ⇐ no stress. mode structure

but

→ Shear flow can induce correlation!

i.e. $$\frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}} (\omega + \underline{k}\cdot\underline{V})$$ (Snell's Law)

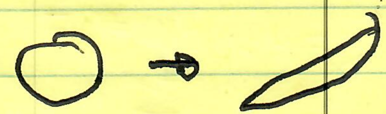
$$\frac{dk_x}{dt} = -\frac{\partial}{\partial x}(k_\theta V_E) = -k_\theta \langle V_E \rangle'$$

$$\boxed{k_x = k_x^{(0)} - k_\theta \langle V_E\rangle' t}$$

~ shearing coordinate

Goldreich
L-B '65

$t \sim \tau_c$



$$k_x = k_x^{(0)} - k_\theta \langle V_E\rangle' \tau_c$$

$$\langle k_x k_y\rangle = \langle k_x^{(0)} k_y\rangle - \boxed{k_\theta^2 \langle V_E\rangle' \tau_c}$$

intrinsic correlation

$\leftrightarrow$ mode structure (esp. radial propagation!)

tilting induced correlation

→ HW: Calculate for slab DW.

Pearlstein
→ Berk

so

$$\langle \tilde v_r \tilde v_\perp\rangle = + \frac{c^2}{B_0^2}\sum_k k_\theta^2 V_E' \tau_{c_k} \cancel{B_0^2} |\phi_k|^2$$

$$\cong \underbrace{\langle \tilde v_r^2\rangle V_E' \tau_{c_k}}_{D}$$

and

$$\partial_t \langle \mathcal{E}_K \rangle \cong -\underbrace{\langle \tilde{v}^2 \rangle \tau_{c,k}}_{\sim -D} \boxed{\langle V_\perp \rangle'^2}$$

note $\sim \langle V_\perp \rangle'^2$! dependence

$$\sim -D \langle V_\perp \rangle'^2$$

- Necessarily, flow eqn. must have same form, opposite sign to conserve energy
- $\langle V_\perp \rangle'^2$ is natural $\mathcal{U}$ variable.

⇒ basic predator-prey structure

⇒ flow - fluctuation kinetic energy exchange

Alternative Perspective:

→ Spectral Energy Evolution

- Recall fluctuations respond adiabatically to Z.F.

i.e. $\omega \sim \omega_*$ vs. $\omega \sim \Omega$

then, have for wave action density

$N = \sum_k \frac{\varepsilon_k}{\omega_k} \rightarrow$ energy density

$\frac{E}{\omega} \rightarrow \ell(t)$

DI²H, (cited)

$$\boxed{\partial_t N + (\underline{v}_{gr} + \underline{V}) \cdot \nabla N - \frac{\partial}{\partial x}(\omega + \underline{k}\cdot\underline{V}) \cdot \frac{\partial N}{\partial \underline{k}} = C(N)}$$

relaxation

$$\partial_t \langle N \rangle + \frac{\partial}{\partial k_r} \left\langle -\frac{\partial}{\partial x}(k_\theta \tilde{V}_E)\, \tilde{N} \right\rangle = C(N)$$

HW → show!

and usual quasilinear crank ⇒ see (DI²H '05)

$$\partial_t \langle N \rangle - \boxed{\partial_{k_r} D_k \partial_{k_r} \langle N \rangle} = \langle C(N) \rangle$$

$$\boxed{D_k = \sum_q \underbrace{k_\theta^2 |\tilde{V}_{E,q}|^2}_{\tilde{v}^2} \, \tau_{c,k}} \rightarrow \left[\begin{array}{l}\text{induced} \\ \text{diffusion}\end{array}\right.$$

random shearing

Note: $\xrightarrow{v_g}$ wave model $\rightarrow \Omega/q$

$$\frac{i}{\Omega - q v_g + i|\gamma|} \longleftrightarrow \frac{i}{\omega - k v}$$

$$\partial_{k_r} D_k \, \partial_{k_r} \sim \frac{k_\theta^3}{k_r^2} \underbrace{\sum_q |\tilde{v}_{Eq}|^2 q^2}_{\langle \tilde{v}^2 \rangle} \tilde{\tau}_c$$

as before.

HW
work out correspondence

then:

$$\langle \mathcal{E} \rangle = \omega \langle N \rangle$$

$$\partial_t \langle \mathcal{E} \rangle = -\int dk \, D_k \frac{\partial}{\partial k_r} \langle N \rangle \frac{\partial \omega_k}{\partial k_r}$$

$$= -\int dk \, \frac{\partial \omega}{\partial k_r} D_k \frac{\partial}{\partial k_r} \langle N \rangle$$

$\frac{\partial \omega}{\partial k_r} < 0$; $D_k \sim \langle \tilde{v}^2 \rangle$; $\frac{\partial \langle N \rangle}{\partial k_r} < 0$

$\frac{\omega}{1 + k_\perp^2 \rho_s^2}$

backward wave

for damping

$\rightarrow$ $\boxed{\text{Same structure}}$ $\checkmark$ $\partial_t \mathcal{E} \approx - \langle \tilde{v}^2 \rangle \mathcal{E} k$

Aside

c.) → $\boxed{\text{Lessons From Ecology}}$ (c.f. May)

→ What can be said about this type of system?

↓

1 predator - 1 prey

Prop.? Solns

General:

$$\frac{dH}{dt} = H \, F(H, P)$$

$$\frac{dP}{dt} = P \, G(H, P)$$

$H \to$ prey

$P \to$ predator

→ Lotka - Volterra (Linear)

breeding

Mathes

$$\frac{dH}{dt} = \gamma H - \alpha HP$$

conservation

$$\frac{dP}{dt} = -dP + \alpha HP$$

↓

depletion

Linear osc

LCO

→ Linear oscillations / growth, damping → show

→ UO → DW − ZF

↓ departure, L-V.

$$\frac{1}{2}\frac{dE}{dt} = \gamma_0 E - \alpha_1 E^2 - \alpha_2 U E$$

DW → ~~[illegible]~~ prey

$$\frac{1}{2}\frac{dU}{dt} = -\mu U + \alpha_2 U E$$

Shear Flow → predator.

→ For 1 pred, 1 prey:

$$\boxed{\begin{aligned} \frac{dH}{dt} &= H\, F(H,P) \\ \frac{dP}{dt} &= P\, G(H,P) \end{aligned}} \quad **$$

⇒ $\boxed{\text{Kolmogorov Thm. (1936), based on Poincaré-Bendixson Theorem.}}$

→ Predator - Prey systems of form (**) have either:

- a stable equilibrium point

or

- a stable limit cycle

if F, G continuous, with continuous first derivatives

characterize sensible syst.

and

(i) $\partial F/\partial P < 0$ → rate of prey ~~increase~~ growth decrease as pred ↑

cs → on renewal

(ii) $H(\partial F/\partial H) + P(\partial F/\partial P) < 0$

→ prey growth rate decreases with population size

cs ↓ resource depletion

(iii) $\partial G/\partial P < 0$

→ rate of increase ~~of~~ predators decreases with population size of pred.

cs ↓ on collapse

iv.) $\boxed{H(\partial G/\partial H) + P(\partial G/\partial P) > 0}$

CS maintain → # predators is increasing function of population size

v.) $F(0, 0) > 0$ $\qquad \frac{dH}{dt} = F H$

CS Prey have positive growth rate (~ linear), for small population $\qquad$ ~~$\gamma + f$~~

and

$\exists$ A, B, C s/t $\qquad \frac{dH}{dt} = H F$

$= H[\gamma - \alpha P - \cancel{\beta H}]$

$\boxed{P = \gamma/\alpha}$

vi.) $F(0, A) = 0$, with $A > 0$

→ can have predator population to stop further prey increase, for low levels prey

(*) → predation induced sat'n.

vii.) $F(B, 0) = 0$, with $B > 0$

→ critical prey population beyond which prey cannot increase, even absent predators!

(*)

$$\frac{\partial H}{\partial t} = H\,F = H\left[\gamma - \alpha_1 P - \beta H\right]$$

$H = \gamma/\beta$

(*) → state prey must be able to self-saturate, as for logistic competition, absent predators $\boxed{B = \gamma_0/\alpha_1}$

vii) $\sigma(C, 0) = 0$, with $C > 0$

There exists a critical prey size C that stops further increase in predators.

$\frac{\partial P}{\partial t} = P\,G = P\left[-\mu + \alpha_2 H\right]$

[redacted] $\left[C \to \mu/\alpha_2\right]$

ix) $\boxed{B > C}$, or system collapses.

[need $\gamma_0 > \alpha_1 \mu/\alpha_2$] or flows collapse.

i) − ix) satisfied ⇒

system will have fixed point or LCO solution structure!

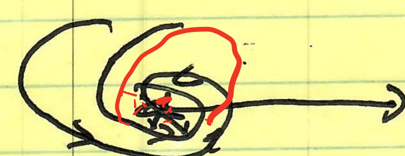
N.B.:

→ i.) → ix.) are criteria for sustainable, 'sensible', system

→ some can be $\geq$, $\leq$

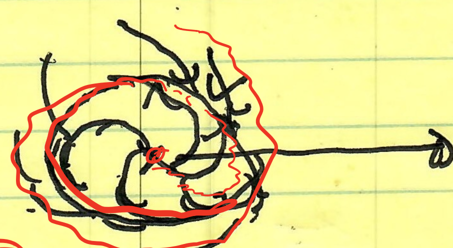
→ Criteria for (stationary state) attractor

i.e. fixed pt



→ fixed pt stable

LCO

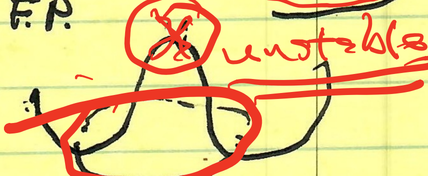


→ fixed pt is unstable.

but LCO encircles it

F.P. unstable

but



i.e. 'Mexican hat' potential

so, if system satisfies K-Thm and fixed pt. unstable, then $\boxed{\text{LCO}}$

LCO → <u>time delay</u>

→ Sense of effect — Model variations starting From Lotka-Volterra:

$$\frac{dH}{dt} = H[a - \alpha P]$$

$$\frac{dP}{dt} = P[-b + \beta H]$$

- all > 0
- often $\alpha = \beta$

1) $a \to r(1 - H/K) + \ldots$ <u>Logistic</u>

↓

<u>carrying capacity</u> (self-satn.)

stabilizing

2) $-\alpha HP$ → prey removal by predator

→ $kP(1-e^{cH})$ destabilizing (saturating)

→ $\dfrac{kHP}{H+D}$ saturation of predation at D

→ $\boxed{\text{Transitions}}$ → K Thm. Conditions

- back to PD '94 model:

Fixed pts:

- trivial: $E = U = 0$

K Thm.

- No flow: $E = \gamma_0/\alpha_1$ → self-soln

  $U = 0$ no flow

('L mode')

- Flow: $E = \mu/\alpha_2$ → turbulence set by flow damping

self:

$$U = \left(\gamma_0 - \frac{\alpha_1 \mu}{\alpha_2}\right)\Big/\alpha_2$$

→ flow set by turb growth

$B > C$

$$\frac{\gamma_0}{\alpha_1} > \frac{\mu}{\alpha_2}$$

Superficially counterintuitive

⇒ signature of pred-prey system.

⇒ observed in GK simulations Lin, et. al. '98

$\mathcal{E} \sim \nu$

Fluctn → Fl. + Flow

Clearly:

- transition for: K Thm ⇒ cond. mult. f.p. transitions

$\gamma_0 > \alpha_1 \mathcal{U}_2$ (CB > C criterion)

Power Threshold

~ P_{thresh}, as $\gamma \sim |\nabla P| \sim Q / D$

⇒ sets critical power, flux

- Can linearize about 2 fixed pts (non-trivial) HW

$0, 0$

No flow:

$(\gamma_0/\alpha_1, 0)$

"mode" → eigenmode of P-P syst.

B > C

slow

$\gamma = -(\mathcal{U}_1 - \alpha_2\gamma_0/\alpha_1) \rightarrow \mathcal{U}$

$\gamma = -\gamma_0 \Rightarrow E$

Observe:

- E mode always heavily damped.
- $\mathcal{U}$ mode 'soft' near transition
  $\gamma \rightarrow 0$ slower

excit. flow

- $\gamma_u \to 0$ at threshold

$\Rightarrow$ $\tau_{transition} \to \infty$ (i.e. a bifurcation).

~ 2nd order transition, soft mode is signature.

Now, since $|\gamma_E| \gg |\gamma_u|$,

- E will relax to equilibria/fixed point much faster than u

$\Rightarrow$

- 'slave' flctns to flow, i.e.

$$\frac{1}{2}\frac{dE}{dt} \to 0 = (\gamma_0 - \alpha_1 E - \alpha_2 u)E$$

$$E = (\gamma_0 - \alpha_2 u)/\alpha_1$$

$$\frac{1}{2}\frac{du}{dt} = -\mu u + \alpha_2 u E$$

$$= -\mu u + \alpha_2 u(\gamma_0 - \alpha_2 u)/\alpha_1$$

$\Rightarrow$, system described by flow:

$$\frac{1}{2}\frac{dU}{dt} = \left(\frac{\alpha_2}{\alpha_1}\gamma_0 - 1\right)U - \frac{\alpha_2^3}{\alpha_1}U^2$$

U_I^{2}

$$\boxed{\frac{1}{2}\frac{dU}{dt} = \left(\frac{\alpha_2}{\alpha_1}\gamma_0 - 1\right)U - \frac{\alpha_2^3}{\alpha_1}U^2}$$

Logistic Eqn.

→ Logistic eqn, with growth threshold

\+ Diffn → Fisher Eqn. → propagating fronts

→ if write for $\langle V_1' \rangle$,

$$\boxed{\frac{d\langle V_1'\rangle}{dt} = \left(\frac{\alpha_2}{\alpha_1}\gamma_0 - 1\right)\langle V_1'\rangle - \frac{\alpha_2^3}{\alpha_1}\langle V_1'\rangle^3}$$

→ TDGL → 2nd order transition

and can add external torque/drive:

$$\frac{d}{dt}\langle v_1 \rangle = \left(\frac{\alpha_2}{\alpha_1}\gamma_0 - \mu\right)\langle v_1' \rangle - \frac{\alpha_2^2}{\alpha_1}\langle v_1' \rangle^3$$

$$+ T_{ext}$$

- Bias? (a la J-TEXT experiments)

Hext

- ala response in ferromagnet near criticality / away from criticality

L+L Stat. Mech

⇒ Topic for serious further work....

Ki Thm → transitions

B > G

→ Furthur:

- LCO's − $\boxed{\text{time delays}}$
  next class.

- $\boxed{\text{model developments}}$

→ Near criticality → $\boxed{\text{noise}}$

$$\frac{1}{2}\frac{du}{dt} = \left(\frac{\alpha_2}{\alpha_1}\gamma_0 - \nu\right)u - \frac{\alpha_2^2}{\alpha_1}u^2$$

$$Q = \bar{Q} + \tilde{Q}$$

$$\gamma_0 \sim \gamma_0(\nabla P) \sim \gamma_0(Q/D)$$

$$\sim \bar{\gamma} + \tilde{\gamma}$$

as $\boxed{Q = \bar{Q} + \tilde{Q}}$

$\bar{Q}$ ↓ mean heatflux

$\tilde{Q}$ ↳ bursts, noise
i.e. pdf edge heat flux

pdf $[v^2]$

viscosity

"Const"

use

not const.

→ Begs the question of

how deal with $\boxed{\text{noise}}$?

⇒ Stochastic model → pdf

↓

Fokker Planck Eqn.

$$D_V = \frac{\beta}{m} V_{Th}^2 \sim \frac{|\tilde{f}_0|^2 \tau_{ac}}{m^2}$$

– Multiplicative Noise (Simple)

Consider Logistic Eqn. → Populations

$$\frac{dN}{dt} = \underbrace{N}_{\substack{\text{Malthusian}\\ \text{growth}\\ \text{(exponential)}}}(k - N)$$

↳ saturation by competition $\sim N^2$

$N \equiv$ # of population

$$x_{n+1} = \alpha x_n (1 - x_n)$$

Logistic Map

$N = 0$, $N = k$ are fixed pts

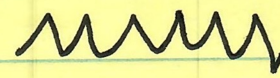
Now, could consider variability in k, and treat as stochastic variable.

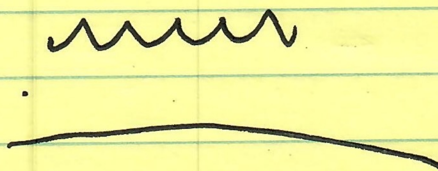
$$\frac{dN}{dt} = N\left(k_0 + \underbrace{\tilde{\gamma}(t)}_{\substack{\text{variability in}\\ \text{resources}}} - N\right) + \underbrace{\tilde{\alpha}(t)}_{\substack{\text{external input}\\ \text{variability}}}$$

⇒ multiplicative noise → rate

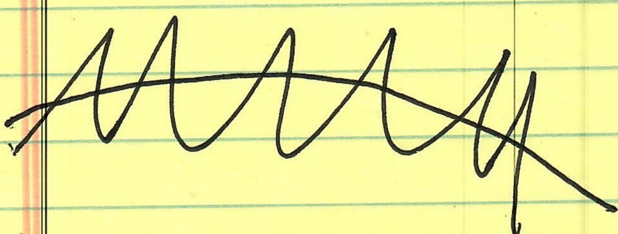
⇒ additive noise.

i.e. additive:

 f + 

noise on top deterministic base = 

multiplicative:



multiplies by fast, random quantity

$f(N,t) \rightarrow$ population pdf

$\downarrow$

How treat?

→ Fokker-Planck Equation for $f(N)$

→ here $\langle \tilde{\gamma}(t)\, \tilde{\gamma}(t') \rangle = |\tilde{\gamma}_0|^2 \tau_{ac}\, \delta(t-t')$

Delta correlated for simplicity.

N.B. This is a "textbook model".

→ additive, as usual

$$\langle \tilde{\alpha} \tilde{\gamma} \rangle \doteq 0$$

Then : $$\frac{dN}{dt} = N\left(\gamma_0 + \tilde{\gamma}(t) - N\right) + \tilde{\alpha}$$

&

$$\frac{\partial}{\partial t} f(N,t) = -\frac{\partial}{\partial N}\left[\left(\gamma_0 N - N^2\right) f(N,t) - \frac{\partial}{\partial N}\left(D\, f(N)\right)\right]$$

For D;

$$\langle \Delta N \, \Delta N \rangle = \int^t dt' \int^t dt'' \, \langle \tilde{\gamma}(t') \tilde{\gamma}(t'') \rangle \, N^2 \quad !$$

$$+ \int^t dt' \int^t dt'' \, \langle \alpha(t') \alpha(t'') \rangle$$

$$= |\gamma_0|^2 \tau_{ac} N^2 t + |\alpha_0|^2 \tau_{ac}$$

Nonlinearity in D

$\rightarrow$ one trademark feature of multiplicative noise

→ Note: $N \to \infty \Rightarrow D \to 0$

Rate variation ⇒ pdf spread

in proportion to population.

→ Additive correction significant at low N.

Now, ignoring additive correction,

$$\boxed{\partial_t f(N) = -\frac{\partial}{\partial N}\left\{ (k_0 N - N^2) f(N,t) - \frac{\partial}{\partial N}\left(\frac{|\tilde{\gamma}_0|^2}{2} \tau_{ac} N^2 f(N,t) \right) \right\}}$$

is Fokker-Planck Equation

and stationarity:

$$N(k_0 - N) f(N) = \frac{\partial}{\partial N}\left(\frac{|\tilde{\gamma}_0|^2}{2} \tau_{ac} N^2 f(N) \right)$$

Norm

so

$$\boxed{f(N) = c \cdot n^{\left[2(k_0/\sigma^2) - 2\right]} e^{-2N/\sigma^2}}$$

$\sigma^2 = |\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}$ Power exponential tail

Need $k_0^2 > (\sigma^2/2)^2 \Leftrightarrow f > 1/n$ $n \to 0$ to avoid log. singularity

i.e. $\boxed{k_0 > \frac{|\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}}{2}}$

Physics of $k_0 > \frac{|\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}}{2}$?

Convenient to linearize around fixed point:

$$\frac{dN}{dt} = (k + \tilde{\gamma} - N)N$$

Validity ?

$$N = k_0 + \tilde{n}$$

$$\frac{d\tilde{n}}{dt} = (k_0 + \tilde{n})(\cancel{k_0} + \tilde{\gamma} - \cancel{k_0} - \tilde{n})$$

$$\cong k_0 \tilde{\gamma} - k_0 \tilde{n} + O(n^2)$$

so

$$\partial_t f(n) = -\frac{\partial}{\partial n}\left[-k_0 n f(n) - \frac{\partial}{\partial n}\left(\frac{k_0^2\gamma_0^2\tau_{ac}}{2} f(n)\right)\right]$$

linearize abt fixed pt.

$$= -\frac{\partial}{\partial n}\left[-k_0 n f(n) - \frac{\partial}{\partial n}\left(\frac{k_0^2}{2}\Gamma^2 f(n)\right)\right]$$

$\Rightarrow$ zero flux / stationarity:

$$\boxed{f(n) = \underbrace{c}_{\text{const}} \exp\left[-n^2/k_0\Gamma^2\right]}$$

Valid for: $\langle(\tilde{n}/N_0)^2\rangle = \langle(\tilde{n}/k_0)^2\rangle < 1$

Now $\langle\tilde{n}^2\rangle = \frac{\Gamma^2 k_0}{2}$

so $\langle(\tilde{n}/k_0)^2\rangle < 1 \Rightarrow \boxed{\frac{\Gamma^2}{2k_0} < 1}$

→ again ; $\underbrace{\sigma^2 < 2k_0}$

i.e. fluctuations small compared
logistic growth.

N.B.:

- can determine time evolution
- can get moments
- spatio-temporal dynamics.